ON SOME POSSIBILITIES OF ACCELERATING ELECTRICALLY CONDUCTING FLUIDS WITH THE USE OF CROSSED MAGNETIC FIELDS

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PMM Vol.25, No.5, 1961, pp. 845-850

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(Received June 25, 1961)

In theoretical investigations attention is devoted chiefly to the use of either crossed magnetic and electric fields [1,2,3,4] or running magnetic fields [5,6,7].

Below we investigate driving an incompressible electrically-conducting fluid by means of constant transverse and time-varying longitudinal magnetic field.

In Section 1 we consider the unsteady flow of a viscous incompressible conducting medium in a plane channel provided with a homogeneous transverse magnetic field, where the motion arises as a result of a variable longitudinal magnetic field penetrating the fluid from the walls. The general solution of the problem is found by means of the Laplace transform. In the case of a linear variation with time of the intensity of the external longitudinal field, a simple formula is found expressing the speed and magnetic and electric field intensities at a cross-section of the channel for the limiting regime of uniform motion of the medium.

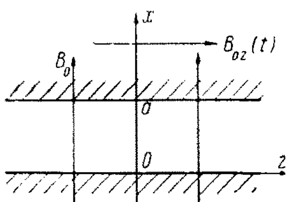

Fig. 1.

A detailed investigation of the transition regime is carried out in Section 2, where the analogous problem is studied for an inviscid fluid. Here it is shown that for sufficiently small values of the magnetic Reynolds number the transition regime has an aperiodic character. The case of uniformly accelerated motion of the medium is also considered.

In Section 3 we investigate driving an inviscid conducting fluid in a channel of annular cross-section.

In view of the complexity of the resulting equations, results are deduced only for the limiting regime of uniform motion.

1. Driving a viscous fluid in a plane channel. We consider the following problem. An infinitely long plane channel of height a is filled with an incompressible viscous electrically conducting fluid. The upper wall of the channel is assumed to be nonconducting and the lower wall ideally conducting. There exists a homogeneous transverse magnetic field B_0 parallel to the x-axis (Fig. 1).

At time $t = 0$ a homogeneous magnetic field $B_{0z}(t)$ is created parallel to the z-axis in the vicinity of the upper wall, whose variation is assumed known. As a result of penetration of this magnetic field into the fluid there is induced in it an electric field E_y and current j_y.

The transverse magnetic field B_0 and current j_y produce a field of body forces, under whose action the conducting fluid is set into motion.

We will assume for simplicity that the physical properties (ρ, η, σ) of the fluid are constant, and that the magnetic permeability μ is equal for the fluid and the walls. We assume also that the displacement current may be neglected (cf. [8], pp. 237, 270).

Then in the region of the upper wall the magnetic field is described by the equations

$$\operatorname{rot} \mathbf{B}^* = 0, \qquad \operatorname{div} \mathbf{B}^* = 0 \qquad (x > a) \tag{1.1}$$

and in the region occupied by the fluid the behavior of the field and the medium is determined by the equations

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \mathbf{j} \times \mathbf{B} + \eta \triangle \mathbf{v}, \qquad \operatorname{div} \mathbf{v} = 0$$
$$\mathbf{j} = \frac{1}{\mu} \operatorname{rot} \mathbf{B} = \sigma (\mathbf{E} + \mathbf{v} \times \mathbf{B}), \qquad \operatorname{div} \mathbf{B} = 0 \tag{1.2}$$
$$\operatorname{rot} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad (0 < x < a)$$

Since in the problem under consideration all quantities depend upon only the one coordinate x, we obtain from the system (1.1) to (1.2) the relations

$$h_x^* = 1, \qquad h_z^* = h_0(\tau) \qquad (\xi > 1) \tag{1.3}$$

$$\frac{\partial u}{\partial \tau} = \frac{\partial h}{\partial \xi} + \frac{1}{R} \frac{\partial^2 u}{\partial \xi^2}, \qquad j = -\frac{\partial h}{\partial \xi} = R_m (e + u), \qquad \frac{\partial h}{\partial \tau} = -\frac{\partial e}{\partial \xi} \qquad (0 < \xi < 1) \tag{1.4}$$

Here dimensionless variables have been introduced according to

$$u = \frac{v_z}{v_a}, \quad \xi = \frac{x}{a}, \quad \tau = \frac{tv_a}{a}, \quad h = \frac{B_z}{B_0} \tag{1.5}$$
$$e = \frac{E_y}{B_0 v_a}, \quad j = \frac{\mu a j_y}{B_0}, \quad R = \frac{\rho v_a a}{\eta}, \quad R_m = \sigma\mu v_a a$$

($v_a = B_0/\sqrt{(\mu\rho)}$ is the speed of Alfven waves)

The system (1.4) is to be solved under the following initial and boundary conditions:

$$u = 0, \quad h = 0, \quad e = 0 \qquad \text{for } \tau = 0 \tag{1.6}$$
$$u = 0, \quad e = 0, \qquad \text{for } \tau > 0, \quad \xi = 0$$
$$u = 0, \quad h = h_0(\tau) \qquad \text{for } \tau > 0, \quad \xi = 1 \tag{1.7}$$

Applying the Laplace transformation to Equations (1.4) we obtain

$$pU = \frac{dH}{d\xi} + \frac{1}{R}\frac{d^2U}{d\xi^2}, \qquad J = -\frac{dH}{d\xi} = R_m(E + U), \qquad pH = -\frac{dE}{d\xi} \tag{1.8}$$

where the boundary conditions (1.7) take the form

$$E = U = 0 \qquad \text{for } \xi = 0$$
$$H = H_0, \quad U = 0 \qquad \text{for } \xi = 1 \tag{1.9}$$

From the system (1.8) we find by elimination the following equations for the transform of the speed:

$$\frac{d^4U}{d\xi^4} - [RR_m + (R + R_m)\,p]\frac{d^2U}{d\xi^2} + RR_m p^2 U = 0 \tag{1.10}$$

where $RR_m = \dfrac{B_0^2 a^2 \sigma}{\eta} = M^2$ (M = Hartmann number)

Solving Equations (1.10) and determining the constants of integration from the boundary conditions (1.9), we obtain the following formulas for the transforms of the speed and field intensities:

$$U = H_0 \frac{\sinh m \sinh n\xi - \sinh n \sinh m\xi}{\left(p - \dfrac{n^2}{R}\right)\dfrac{\sinh m \cosh n}{n} - \left(p - \dfrac{m^2}{R}\right)\dfrac{\sinh n \cosh m}{m}} \tag{1.11}$$

$$H = H_0 \frac{\left(p - \dfrac{n^2}{R}\right)\dfrac{\sinh m \cosh n\xi}{n} - \left(p - \dfrac{m^2}{R}\right)\dfrac{\sinh n \cosh m\xi}{m}}{\left(p - \dfrac{n^2}{R}\right)\dfrac{\sinh m \cosh n}{n} - \left(p - \dfrac{m^2}{R}\right)\dfrac{\sinh n \cosh m}{m}} \tag{1.12}$$

$$E = -pH_0 \frac{\left(p - \frac{n^2}{R}\right)\frac{\sinh m \sinh n\xi}{n^2} - \left(p - \frac{m^2}{R}\right)\frac{\sinh n \sinh m\xi}{m^2}}{\left(p - \frac{n^2}{R}\right)\frac{\sinh m \cosh n}{n} - \left(p - \frac{m^2}{R}\right)\frac{\sinh n \cosh m}{m}} \tag{1.13}$$

where (1.14)

$$m = \frac{1}{2}\left[\sqrt{RR_m + (\sqrt{R} + \sqrt{R_m})^2 p} + \sqrt{RR_m + (\sqrt{R} - \sqrt{R_m})^2 p}\right]$$
$$n = \frac{1}{2}\left[\sqrt{RR_m + (\sqrt{R} + \sqrt{R_m})^2 p} - \sqrt{RR_m + (\sqrt{R} - \sqrt{R_m})^2 p}\right]$$

The transformation is inverted using the inversion formulas

$$u = \frac{1}{2\pi i}\int_L U \exp p\tau\, dp, \quad h = \frac{1}{2\pi i}\int_L H \exp p\tau\, dp, \quad e = \frac{1}{2\pi i}\int_L E \exp p\tau\, dp \tag{1.15}$$

We restrict ourselves further to the case when the intensity of the external magnetic field varies according to the law $h_0 = a\tau$ ($H_0 = a/p^2$). Since U, H and E are then functions of p whose sign does not change, their inversions are found by summation of the residues at the poles $p = 0$, $p = p_n$, where the p_n are the roots of the denominator in Equations (1.11) to (1.13).

An investigation of these roots was carried out by the authors in [9], where an approximate expression was obtained for them. Here we give only the formulas for the limiting regime of uniform motion of the fluid, which are found by calculating the residues at the pole $p = 0$:

$$\frac{u_0}{\alpha} = \frac{\xi \sinh M - \sinh M\xi}{\sinh M}, \qquad \frac{h_0}{\alpha} = \tau - \frac{M(\cosh M - \cosh M\xi)}{R \sinh M} \tag{1.16}$$
$$\frac{e_0}{\alpha} = -\xi, \qquad \frac{j_0}{\alpha} = -R_m \frac{\sinh M\xi}{\sinh M}$$

Thus by means of crossed magnetic fields – a uniform transverse one and a longitudinal one that increases linearly with time – it is possible to create a uniformly moving stream of fluid.

2. Driving an ideal fluid in a plane channel. We consider the problem posed in Section 1 for the case of an inviscid electrically conducting fluid. With $\eta = 0$ the system of equations (1.8) takes the form

$$pU = \frac{dH}{d\xi}, \qquad J = -\frac{dH}{d\xi} = R_m(E + U), \qquad pH = -\frac{dE}{d\xi} \tag{2.1}$$

where the boundary conditions are the following:

$$E = 0 \quad \text{for } \xi = 0, \qquad H = H_0 \quad \text{for } \xi = 1 \tag{2.2}$$

From (2.1) we find the equation

$$\frac{d^2H}{d\xi^2} - \frac{p^2}{1 + p / R_m} H = 0 \tag{2.3}$$

whose solution we write in the form

$$H = c_1 \cosh \frac{p\xi}{\sqrt{1 + p / R_m}} + C_2 \sinh \frac{p\xi}{\sqrt{1 + p / R_m}} \tag{2.4}$$

Finding expressions for U and E from (2.1) with the use of (2.4), and determining the constants of integration by means of the boundary conditions (2.2), we obtain the following equations for U, H and E:

$$U = \frac{H_0}{\sqrt{1 + p / R_m}} \frac{\sinh \vartheta\xi}{\cosh \vartheta}, \qquad H = H_0 \frac{\cosh \vartheta\xi}{\cosh \vartheta} \tag{2.5}$$

$$E = - H_0 \sqrt{1 + p / R_m}\, \frac{\sinh \vartheta\xi}{\cosh \vartheta} \qquad \left(\vartheta = \frac{p}{\sqrt{1 + p / R_m}}\right)$$

Let $h_0 = \alpha\tau$ as above. Then the determination of u, h and e is reduced to the calculation of the residues at the poles

$$p = 0, \; p_{1k} = - \frac{\lambda_k^2}{2R_m} + \frac{\lambda_k}{2R_m} \sqrt{\lambda_k^2 - 4R_m^2} \tag{2.6}$$

$$p_{2k} = - \frac{\lambda_k^2}{2R_m} - \frac{\lambda_k}{2R_m} \sqrt{\lambda_k^2 - 4R_m^2} \qquad \left(\lambda_k = \frac{2k+1}{2}\pi\right)$$

Performing the calculations, we obtain for the distributions of speed and magnetic and electric field intensities in the fluid

$$\frac{u}{\alpha} = \xi - 2 \sum_{k=0}^{\infty} (-1)^k \frac{\sin \lambda_k \xi}{\lambda_k} \exp\left(-\frac{\lambda_k^2 \tau}{2R_m}\right) \times$$
$$\times \left[\frac{1}{\sqrt{\lambda_k^2 - 4R_m^2}} \sinh\left(\frac{\lambda_k \tau}{2R_m}\sqrt{\lambda_k^2 - 4R_m^2}\right) + \frac{1}{\lambda_k} \cosh\left(\frac{\lambda_k \tau}{2R_m}\sqrt{\lambda_k^2 - 4R_m^2}\right)\right] \tag{2.7}$$

$$\frac{h}{\alpha} = \tau - 4R_m \sum_{k=0}^{\infty} (-1)^k \frac{\cos \lambda_k \xi}{\lambda_k^2} \exp\left(-\frac{\lambda_k^2 \tau}{2R_m}\right) \times$$
$$\times \frac{1}{\sqrt{\lambda_k^2 - 4R_m^2}} \sinh\left(\frac{\lambda_k \tau}{2R_m}\sqrt{\lambda_k^2 - 4R_m^2}\right) \tag{2.8}$$

$$\frac{e}{\alpha} = -\xi + 2 \sum_{k=0}^{\infty} (-1)^k \frac{\sin \lambda_k \xi}{\lambda_k} \exp\left(-\frac{\lambda_k^2 \tau}{2R_m}\right) \left[\frac{1}{\lambda_k} \cosh\left(\frac{\lambda_k \tau}{2R_m}\sqrt{\lambda_k^2 - 4R_m^2}\right) - \right.$$
$$\left. - \frac{1}{\sqrt{\lambda_k^2 - 4R_m^2}} \sinh\left(\frac{\lambda_k \tau}{2R_m}\sqrt{\lambda_k^2 - 4R_m^2}\right)\right] \tag{2.9}$$

The first terms in Equations (2.7) to (2.9) represent the limiting regime of uniform motion of the fluid, and the remaining terms the transitional regime. It follows from these results that for $R_m < \pi/4$ the transitional regime has an aperiodic character, whereas for $R_m > \pi/4$ it exhibits a finite number of damped oscillations.

It may be noted that here, in contrast with a viscous fluid, the current density in the fluid is equal to zero for the regime of established motion.

We consider also the case that the intensity of the external magnetic field increases according to the law $h_0 = \beta\tau^2$.

For the limiting regime we obtain the relationships

$$\frac{u_0}{\beta} = 2\left(\tau - \frac{1}{R_m}\right)\xi, \qquad \frac{h_0}{\beta} = \tau^2 - 1 + \xi^2, \qquad \frac{e_0}{\beta} = -2\xi\tau, \qquad \frac{j_0}{\beta} = -2\xi \tag{2.10}$$

As is seen from Equations (2.10), in this case there is created with the aid of crossed magnetic fields a uniformly accelerated stream of fluid in the channel.

In concluding this section we note the following circumstance arising from Equations (2.7) to (2.9): in the case of plane unsteady motion of an ideal incompressible fluid in a constant transverse magnetic field, the speed, and also the induced electric and magnetic fields, are represented by functions of the form*

$$f = \exp\left[i\lambda\xi - \frac{\lambda^2\tau}{2R_m} \pm \frac{\lambda\tau}{2R_m}\sqrt{\lambda^2 - 4R_m^2}\right] \tag{2.11}$$

(where λ is an arbitrary parameter), satisfying the equation

$$\frac{\partial^2 f}{\partial\xi^2} + \frac{1}{R_m}\frac{\partial^2 f}{\partial\xi\partial\tau} - \frac{\partial^2 f}{\partial\tau^2} = 0 \tag{2.12}$$

Solutions of this equation for the case of a half-space were considered in [10].

3. Driving an ideal fluid in a channel of annular cross-section. We investigate now the possibility of driving an inviscid conducting fluid in an infinitely long channel of annular cross-section, which is formed by an inner ideally conducting cylinder of radius a and an outer nonconducting cylinder of radius b. There is a radial magnetic field $B_r = B_0 a/r$. Motion of the fluid arises as a result of penetration

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* For $R_m \to \infty$ these solutions become Alfven waves.

of a uniform longitudinal magnetic field $B_z(t)$, which is created in the vicinity of the outer cylinder by external annular currents.

The investigation of unsteady longitudinal motion of a viscous conducting fluid in an annular channel in the presence of a radial magnetic field presents significant mathematical difficulty. In [11] it was found possible to integrate the appropriate equations only in the special case of equal viscous and magnetic Reynolds numbers. We therefore restrict consideration to the flow of an inviscid fluid. For the region occupied by the fluid we have the following equations:

$$\frac{\partial u}{\partial \tau} = -\frac{1}{\xi} j, \qquad j = -\frac{\partial h}{\partial \xi} = R_m\left(e + \frac{u}{\xi}\right), \qquad \frac{1}{\xi}\frac{\partial}{\partial \xi}(\xi e) = -\frac{\partial h}{\partial \tau} \tag{3.1}$$

Here

$$u = \frac{v_z}{v_a}, \qquad \xi = \frac{r}{a}, \qquad \tau = \frac{t v_a}{a}, \qquad h = \frac{B_z}{B_0} \tag{3.2}$$

$$e = \frac{E_\varphi}{B_0 v_a}, \qquad j = \frac{j_\varphi \mu a}{B_0}, \qquad v_a = \frac{B_0}{\sqrt{\mu\rho}}, \qquad R_m = \sigma\mu v_a a$$

The initial and boundary conditions for the problem have the form

$$u = 0, \qquad h = 0, \qquad e = 0 \qquad \text{for } \tau = 0 \tag{3.3}$$

$$e|_{\xi=1} = 0, \quad h|_{\xi=\xi_0} = h_0(\tau) \qquad \text{for } \tau > 0 \qquad \left(\xi_0 = \frac{b}{a}\right) \tag{3.4}$$

Applying the Laplace transformation, we obtain from (3.1)

$$pU = -\frac{1}{\xi} J, \qquad J = -\frac{dH}{d\xi} = R_m\left(E + \frac{U}{\xi}\right), \qquad \frac{1}{\xi}\frac{d}{d\xi}(\xi E) = -pH \tag{3.5}$$

From this system we find for the transform H

$$\left(1 + \frac{R_m}{\xi^2 p}\right)\frac{d^2H}{d\xi^2} + \frac{1}{\xi}\left(1 - \frac{R_m}{\xi^2 p}\right)\frac{dH}{d\xi} - R_m p H = 0 \tag{3.6}$$

Substituting $x = R_m\sqrt{(1 + p\xi^2/R_m)}$, we obtain the Bessel equation

$$\frac{d^2H}{dx^2} + \frac{1}{x}\frac{dH}{dx} - H = 0 \tag{3.7}$$

whose solution has the form

$$H = C_1 I_0(x) + C_2 K_0(x) \tag{3.8}$$

Determining C_1 and C_2 from the transformed boundary conditions

$$\frac{dH}{dx} = 0 \quad \text{for } x = x_1 = R_m\left(1 + \frac{p}{R_m}\right)^{1/2}, \quad H = H_0 \quad \text{for } x = x_2 = R_m\left(1 + \frac{p\xi_0}{R_m}\right)^{1/2} \tag{3.9}$$

we obtain the following expressions for the transforms U, H, E:

$$U = \frac{R_m H_0}{x\Delta} [I_1(x) K_1(x_1) - I_1(x_1) K_1(x)]$$
$$H = \frac{H_0}{\Delta} [I_0(x) K_1(x_1) + I_1(x_1) K_0(x)] \qquad (3.10)$$
$$E = -\frac{H_0 x \sqrt{p}}{\Delta \sqrt{R_m (x^2 - R_m^2)}} [I_1(x) K_1(x_1) - I_1(x_1) K_1(x)]$$

where

$$\Delta = I_0(x_2) K_1(x_1) + I_1(x_1) K_0(x_2) \qquad (3.11)$$

Leaving aside the investigation of the transitional regime, we limit ourselves to giving the formulas for the limiting regime of uniform motion of the medium obtained in the case $h_0 = \alpha\tau$:

$$\frac{u_0}{\alpha} = \frac{1}{2}(\xi^2 - 1), \qquad \frac{h_0}{\alpha} = \tau, \qquad \frac{e_0}{\alpha} = -\frac{1}{2\xi}(\xi^2 - 1) \qquad (3.12)$$

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Translated by M.D.v.D.